AD-A103 881   WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER   F/G 12/1
AN INTEGRAL OF PRODUCTS OF LEGENDRE FUNCTIONS AND A CLEBSCH-GOR--ETC(U)
JUL 81   R ASKEY   DAAG29-80-C-0041
UNCLASSIFIED   MRC-TSR-2250   NL

END
DATE
FILMED
0 81
DTIC

MRC Technical Summary Report # 2250

AN INTEGRAL OF PRODUCTS OF LEGENDRE FUNCTIONS AND A CLEBSCH-GORDAN SUM

Richard Askey

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

July 1981

(Received July 2, 1981)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

AN INTEGRAL OF PRODUCTS OF LEGENDRE FUNCTIONS AND A CLEBSCH-GORDAN SUM

Richard Askey*

Technical Summary Report #2250

July 1981

ABSTRACT

New proofs and extensions are given of a sum considered by A. M. Din involving Clebsch-Gordan coefficients with zero magnetic quantum numbers and of an integral involving the product of three Legendre functions, one of the second kind.

AMS (MOS) Subject Classifications: 33A45, 33A70, 81.33

Key Words: Legendre polynomials, Legendre function, Clebsch-Gordan coefficients

Work Unit Number 1 - Applied Analysis

A

* Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This work was partially supported by the National Science Foundation under Grant No. MCS-8101568.

AN INTEGRAL OF PRODUCTS OF LEGENDRE FUNCTIONS AND A CLEBSCH-GORDAN SUM

Richard Askey*

Din [1] showed that

$$S := \sum_{\substack{i=|c-b| \\ i\neq a}}^{c+b} \frac{2i+1}{i(i+1) - a(a+1)} \left(C^{c0}_{i0b0}\right)^2 = 0 \tag{1}$$

when a, b and c are non-negative integers with $a + b + c$ odd and $|c-b| \leq a \leq c+b$. The Clebsch-Gordan coefficients with zero magnetic quantum numbers are given by

$$\left(C^{c0}_{i0b0}\right)^2 = \frac{2c+1}{2} \int_{-1}^{1} dx P_i(x) P_b(x) P_c(x) \quad , \tag{2}$$

This integral was evaluated by Ferrers and others in the last century. The evaluation comes from the linearization formula

$$P_n(x)P_m(x) =$$

$$\sum_{k=0}^{\min(m,n)} \frac{(1/2)_{m-k}(1/2)_{n-k}(1/2)_k (m+n-k)!(m+n-2k+1/2)}{(m-k)!(n-k)!k!(1/2)_{m-n-k}(m+n-k+1/2)} P_{m+n-2k}(x) \quad , \tag{3}$$

and the orthogonality of Legendre polynomials. See [2]. To show (1) Din reduced it to showing that

$$I(a,b,c) := \int_{-1}^{1} dx\, Q_a(x) P_b(x) P_c(x) = 0 \tag{4}$$

when $a, b, c \geq 1$ are integers, $a + b + c$ is odd and $|c-a| < b < c+a$. Here $P_i(x)$ is the Legendre polynomial and $Q_a(x)$ is the Legendre function of the second kind on the cut $[-1,1]$. He ended the paper by stating that I could evaluate (4) for general integers a, b, c. The details follow.

---

* Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This work was partially supported by the National Science Foundation under Grant No. MCS-8101568.

Din started with

$$\int_{-1}^{1} dx\, Q_a(x)P_b(x) = \frac{1-\cos(b-a)\pi}{(b-a)(b+a+1)},\ a,b = 1,2,\ldots,\ a \neq b\ , \qquad (5)$$

with a reference to [3]. A generalization of (5) is given there when a and b are complex, $\mathrm{Re}\ a > 0$, $\mathrm{Re}\ b > 0$, and the extra term which occurs vanishes when either a or b is an integer. The argument in [3] used the Legendre differential equation. Here is a second derivation of (5). Start with an expansion of Heine [4]

$$Q_a(\cos\theta) = \frac{2\ a!}{(\frac{3}{2})_a} \sum_{i=0}^{\infty} \frac{(\frac{1}{2})_i (a+1)_i}{i!\,(a+\frac{3}{2})_i} \cos(a+2i+1)\theta\ .$$

The shifted factorial $(c)_n$ is defined by

$$(c)_n = \Gamma(n+c)/\Gamma(c) = c(c+1)\cdots(c+n-1)\ .$$

Since $P_a(-x) = (-1)^a P_a(x)$ and $Q_a(-x) = (-1)^{a+1} Q_a(x)$, $a = 0,1,\ldots,$ we may assume a and b have opposite parity, for the integral in (5) vanishes when a and b have the same parity. Then

$$I(a,b,0) = \frac{2\ a!}{(\frac{3}{2})_a} \sum_{i=0}^{\infty} \frac{(\frac{1}{2})_i (a+1)_i}{i!\,(a+\frac{3}{2})_i} \int_0^{\pi} d\theta \cos(a+2i+1)\theta \sin\theta\, P_b(\cos\theta)$$

$$= \frac{2\ a!}{(\frac{3}{2})_a} \sum_{i=0}^{\infty} \frac{(\frac{1}{2})_i (a+1)_i (a+2i+1)(i+(a+b-1)/2)!\,(-\frac{1}{2})_{i+(a+1-b)/2}}{i!\,(a+\frac{3}{2})_i (\frac{3}{2})_{i+(a+b+1)/2} (i+(a+1-b)/2)!}$$

by a special case of an integral of Gegenbauer which is equivalent to [5]

$$C_n^{\mu}(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(\mu)_{n-k} (\mu-\lambda)_k (n-2k+\lambda)}{(\lambda+1)_{n-k}\, k!\ \lambda} C_{n-2k}^{\lambda}(x)\ ,$$

where $C_n^{\lambda}(x)$ is the ultraspherical polynomial.

The above sum can be written as a generalized hypergeometric series and then summed by a formula of Dougall [6]. A more general sume of Dougall will be stated below. A routine reduction shows that (5) holds when $a,b = 0,1,\ldots,$ with the integral equal to zero when $a = b$.

To compute the evaluation of (4) use the Ferrers-Adams linearization formula (3) and (5) to obtain

$$I(a,b,c) = \sum_{k=0}^{\min(b,c)} \frac{(1/2)_{b-k}(1/2)_{c-k}(1/2)_k (b+c-k)!(b+c-2k+1/2)}{(b-k)!(c-k)!k!(1/2)_{b+c-k}(b+c-k+1/2)} ,$$

$$\frac{[1-\cos(b+c-2k-a)\pi]}{(b+c-a-2k)(b+c+a+1-2k)}$$

$$= \frac{[1-\cos(b+c-a)\pi](1/2)_b(1/2)_c(b+c)!}{(b+c-a)(b+c+a+1)b!c!(1/2)_{b+c}}$$

$${}_7F_6\left(\begin{matrix}-b-c-1/2, -b/2-c/2+\frac{3}{4}, -b, -c, 1/2, (a-b-c)/2, (-1-a-b-c)/2 \\ -b/2-c/2-\frac{1}{4}, 1/2-c, 1/2-b, -b-c, (1-a-b-c)/2, (2+a-b-c)/2\end{matrix} ; 1\right)$$

Dougall's sum of the very well poised 2-balanced ${}_7F_6$ [7],

$$\begin{aligned} &{}_7F_6\left(\begin{matrix}a, 1+a/2, b, c, d, e, -n \\ a/2, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n\end{matrix} ; 1\right) \\ &= \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-c-d)_n} \end{aligned} \tag{8}$$

when $1+2a = b+c+d+e-n$, can be used and the result is

$$\begin{aligned} &\int_{-1}^{1} dx\, Q_a(x)P_b(x)P_c(x) \\ &= \frac{[1-\cos(b+c-a)\pi](-(b+c+a)/a)_c((b-c-a+1)/2)_c}{(b+c-a)(b+c+a+1)(-(b+c+a-1)/2)_c((b-c-a)/2)_c} \end{aligned} \tag{9}$$

when $0 \leq b \leq c$, $a+b+c$ odd, and zero when $a+b+c$ is even. Since this integral vanishes when $b+c+a$ is even, we may write $a = b+c+1+2k$. The integral is then

$$\int_{-1}^{1} dx\, Q_{b+c+1+2k}(x)\ P_b(x)P_c(x) \tag{10}$$

$$= -\frac{\Gamma(k+b+c+\frac{3}{2})\Gamma(k+b+1)\Gamma(k+c+1)\Gamma(k+\frac{1}{2})}{2\Gamma(k+b+c+2)\Gamma(k+b+\frac{3}{2})\Gamma(k+c+\frac{3}{2})\Gamma(k+1)} .$$

This integral vanishes when $k = -1,-2,\ldots,-\min(b,c)$ as was shown by Din.

Since (5) holds when a is not an integer, and the rest of the above argument only used the integrality of b and c, formula (8) continues to hold when $\operatorname{Re} a \geqq 0$. In this case it is better to write it as

$$\int_{-1}^{1} dx\, Q_a(x)P_b(x)P_c(x) = \frac{[1-\cos(b+c-a)\pi]\cdot\Gamma(\frac{c-b-a}{2})\Gamma(\frac{b-c-a}{2})}{(b+c-a)(b+c+a+1)\Gamma(\frac{c-b-a+1}{2})\Gamma(\frac{b-c-a+1}{2})}$$

$$\frac{\Gamma(\frac{b+c-a+1}{2})\Gamma(\frac{-b-c-a+1}{2})}{\Gamma(\frac{b+c-a}{2})\Gamma(\frac{-b-c-a}{2})}\ ,\quad \operatorname{Re} a \geqq 0,\ b,c = 0,1,\ldots, \tag{11}$$

with an appropriate limit taken when one of the gamma functions has a pole.

The sum in (1) can be evaluated in exactly the same way, only the details are easier. One only needs to use (2) to replace the Clebsch-Gordan coefficients by a known integral, rewrite the series as a generalized hypergeometric series and use Dougall's sum (8). Fortunately Din was unaware of Dougall's sum, for the integral in (11) seems to be a fundamental result, and it does not seem to have been evaluated before. I was surprised by this, since Hobson [8] wrote that F. E. Neumann had evaluated this integral. However it is not given in the book of Neumann that Hobson mentions nor in the other book of Neumann that I have looked at.

ACKNOWLEDGEMENT

This work was partly supported by NSF Grant No. MCS-8101568. I would like to thank A. M. Din for bringing these problems to my attention.

REFERENCES

1. Din, A. M., Letters in Math. Phys. 5, 207 (1981).
2. Szegő, G., Orthogonal Polynomials, Amer. Math. Soc., Providence, RI (1975), problem 84.
3. Erdélyi, A. et al, Higher Transcendental Functions, Vol. 1, Mc-Graw Hill, New York, 1953 (3.12 (13)).
4. Szegő, G., op. cit. (4.9.16).
5. ibid. (4.10.27)
6. Bailey, W. N., Generalized Hypergeometric Series, Hafner, New York, 1972, 4.4(1).
7. ibid, 4.3(5).
8. Hobson, E. W., The Theory of Spherical and Ellipsoidal Harmonics, Chelsea, New York, 1955, p. 84.

RA/jvs

14 MRC-TSR

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2250	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) An Integral of Products of Legendre Functions and a Clebsch-Gordan Sum		5. TYPE OF REPORT & PERIOD COVERED Summary Report, - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Richard Askey		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS-8101568
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS (see Item 18 below)		12. REPORT DATE July 1981
		13. NUMBER OF PAGES 5
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office and National Science Foundation
P. O. Box 12211 Washington, DC 20550
Research Triangle Park
North Carolina 27709

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Legendre polynomials, Legendre function, Clebsch-Gordan coefficients

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

New proofs and extensions are given of a sum considered by A. M. Din involving Clebsch-Gordan coefficients with zero magnetic quantum numbers and of an integral involving the product of three Legendre functions, one of the second kind.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)